

Relative Motion

INTRODUCTION :

Motion of a body can be observed, when it changes its position with respect to some other body. In this sense, motion is a relative concepts to analyze motion of a body say A, therefore we have to fix our reference frame to some other body say B. The result obtained is motion of body A relative to body B.

- (a) Motion is always relative, there is no such thing like absolute motion or absolute rest.
- (b) Motion is always defined with respect to an observer or reference frame.

REFERENCE FRAME:

Reference frame is an axis system from which motion is observed along with a clock attached to the axis, to measuring time, Reference frame can be stationary or moving.

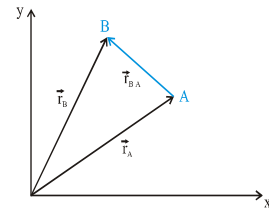
Suppose there are two persons A and B sitting in a car moving at constant speed. Two stationary person C and D observe them from the ground.



Here B appears to be moving for C and D, but at rest for A, similarly C appears to be at rest for D but moving backward for A and B.

RELATIVE POSITION, RELATIVE VELOCITY AND RELATIVE ACCELERATION:

Let two bodies represented by particle A and B at position defined by position vector \vec{r}_A and \vec{r}_B moving with velocities \vec{v}_A and \vec{v}_B and acceleration \vec{a}_A and \vec{a}_B with respect to a reference frame S. For analyzing motion of terrestrial bodies the reference frame S is fixed with the ground.



Relative Position:

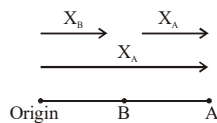
The vector $\vec{r}_{B/A}$ denotes position vector of B relative to A following triangle law of vector addition, we have

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \quad \dots(i)$$

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

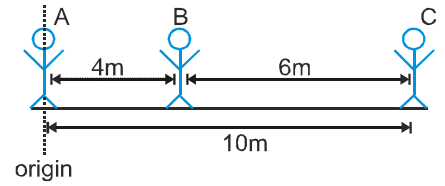
Note: In general if position of A w.r.t. to origin is X_A and that of B w.r.t. origin is X_B then “Position of A w.r.t. B”

X_{AB} is
 $X_{AB} = X_A - X_B$



Ex. See the figure (take +ve direction towards right and -ve towards left)
Find x_{BA} , x_{CA} , x_{CB} , x_{AB} and x_{AC} .

Sol. Here, Position of B w.r.t. A is 4 m towards right. ($x_{BA} = +4\text{m}$)
Position of C w.r.t. A is 10 m towards right. ($x_{CA} = +10\text{m}$)
Position of C w.r.t. B is 6 m towards right ($x_{CB} = +6\text{m}$)
Position of A w.r.t. B is 4 m towards left. ($x_{AB} = -4\text{m}$)
Position of A w.r.t. C is 10 m towards left. ($x_{AC} = -10\text{m}$)



Relative Velocity:

First derivatives of \vec{r}_A and \vec{r}_B with respect to time equals to velocity of particle A and velocity of particle B relative to frame S and first derivative of $\vec{r}_{B/A}$ with respect to time defines velocity of B relative to A

$$V_A = \text{velocity of A w.r.t. ground} = \frac{dx_A}{dt}$$

$$V_B = \text{velocity of B w.r.t. ground} = \frac{dx_B}{dt}$$

$$V_{BA} = \text{velocity of A w.r.t. B} = \frac{dr_{BA}}{dt} = \frac{d}{dt}(r_B - r_A) = \left(\frac{dr_B}{dt} - \frac{dr_A}{dt}\right)$$

$$\vec{V}_B = \vec{V}_A + \vec{V}_{B/A} \Rightarrow \vec{V}_{B/A} = \vec{V}_B - \vec{V}_A$$

Notes :

- All velocities are relative & have no significance unless observer is specified, However when we say “velocity of A”, what we mean, is velocity of A w.r.t. ground which is assumed to be at rest.
- Velocity of an object w.r.t. itself is always zero

Ex. An object A is moving with 5 m/s and B is moving with 20 m/s in the same direction. (Positive x-axis)

(i) Find velocity of B with respect to A.

(ii) Find velocity of A with respect to B

Sol. (i) $v_B = +20\text{ m/s}$ $v_A = +5\text{ m/s}$ $v_{BA} = v_B - v_A = +15\text{ m/s}$

(ii) $v_B = +20\text{ m/s}$, $v_A = +15\text{ m/s}$; $v_{AB} = v_A - v_B = -15\text{ m/s}$

Note :

$$v_{BA} = -v_{AB}$$

Ex. Two objects A and B are moving towards each other with velocities 10 m/s and 12 m/s respectively as shown.



(i) Find the velocity of A with respect to B.

(ii) Find the velocity of B with respect to A

Sol. $v_A = +10$, $v_B = -12$

(i) $v_{AB} = v_A - v_B = (10) - (-12) = 22\text{ m/s}$.

(ii) $v_{BA} = v_B - v_A = (-12) - (10) = -22\text{ m/s}$.

Relative Acceleration:

Second derivatives of \vec{r}_A and \vec{r}_B with respect to time equals to acceleration of particle A and acceleration of particle B relative to frame and second derivative of $\vec{r}_{B/A}$ with respect to time defines acceleration of B relative to A

$$\vec{r}_A = \frac{d\vec{V}_A}{dt}, \quad \vec{r}_B = \frac{d\vec{V}_B}{dt}, \quad \vec{r}_{B/A} = \frac{d\vec{V}_{B/A}}{dt}$$

So $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

- Equation of motion (relative acceleration is constant)

$$V_{rel} = u_{rel} + a_{rel} t$$

$$S_{rel} = u_{rel} t + \frac{1}{2} a_{rel} t^2$$

$$V_{rel}^2 = u_{rel}^2 + 2a_{rel} S_{rel}$$

Velocity of Approach/ separation:

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, we say it is velocity of separation.

In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach/ separation is simply equal to magnitude of relative velocity of A w.r.t. B.

- $a_{app} = \left(\frac{d}{dt}\right) V_{app}, \quad a_{sep} = \frac{d}{dt} V_{sep}$

$$V_{app} = \int a_{app} dt, \quad V_{sep} = \int a_{sep} dt$$

- Ex.** A particle A is moving with a speed of 10 m/s towards right and another particle B is moving at speed of 12 m/s towards left. Find their velocity of approach.



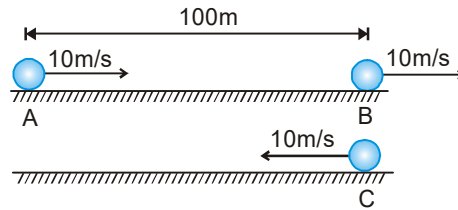
- Sol.** $V_A = +10, V_B = -12 \Rightarrow V_{AB} = V_A - V_B \Rightarrow 10 - (-12) = 22 \text{ m/s}$
 since separation is decreasing hence $V_{app} = |V_{AB}| = 22 \text{ m/s}$ **Ans.:** 22 m/s

- Ex.** A particle A is moving with a speed of 20 m/s towards right and another particle B is moving at a speed of 5 m/s towards right. Find their velocity of approach.



- Sol.** $V_A = +20, V_B = +5 \Rightarrow V_{AB} = V_A - V_B \Rightarrow 20 - (+5) = 15 \text{ m/s}$
 since separation is decreasing hence $V_{app} = |V_{AB}| = 15 \text{ m/s}$ **Ans.** 15 m/s

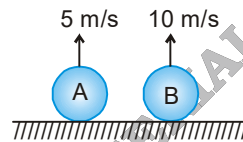
Ex. A particle A is moving with a speed of 10 m/s towards right, particle B is moving at a speed of 10 m/s towards right and another particle C is moving at speed of 10 m/s towards left. The separation between A and B is 100 m. Find the time interval between C meeting B and C meeting A.



Sol. $t = \frac{\text{separation between A and C}}{V_{\text{app of A and C}}} = \frac{100}{10 - (-10)} = 5 \text{ sec.}$ **Ans. :** 5 sec.

Ex. A and B are thrown vertically upward with velocity, 5 m/s and 10 m/s respectively ($g = 10 \text{ m/s}^2$). Find separation between them after one second

Sol. $S_A = ut - \frac{1}{2}gt^2$
 $= 5t - \frac{1}{2} \times 10 \times t^2 = 5 \times 1 - 5 \times 1^2 = 5 - 5 = 0$



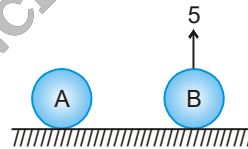
$S_B = ut - \frac{1}{2}gt^2 = 10 \times 1 - \frac{1}{2} \times 10 \times 1^2 = 10 - 5 = 5$

$\therefore S_B - S_A = \text{separation} = 5 \text{ m.}$

Aliter

$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = (-10) - (-10) = 0$

Also $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 10 - 5 = 5 \text{ m/s}$

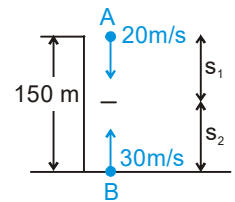


$\therefore \vec{s}_{BA} \text{ (in 1 sec)} = \vec{v}_{BA} \times t$

$= 5 \times 1 = 5 \text{ m}$

\therefore Distance between A and B after 1 sec = 5 m.

Ex. A ball is thrown downwards with a speed of 20 m/s from the top of a building 150 m high and simultaneously another ball is thrown vertically upwards with a speed of 30 m/s from the foot of the building. Find the time after which both the balls will meet. ($g = 10 \text{ m/s}^2$)



Sol. $S_1 = 20t + 5t^2 \Rightarrow S_2 = 30t - 5t^2 \Rightarrow S_1 + S_2 = 150$
 $\Rightarrow 150 = 50t \Rightarrow t = 3 \text{ s}$

Aliter

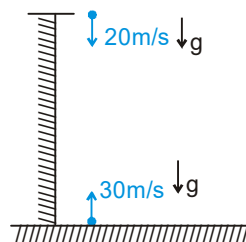
Relative acceleration of both is zero since both have same acceleration in downward direction

$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = g - g = 0$

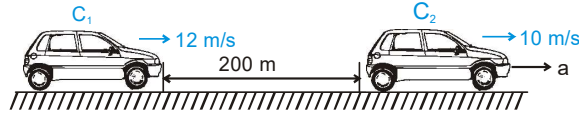
$\vec{v}_{BA} = 30 - (-20) = 50$

$s_{BA} = v_{BA} \times t$

$t = \frac{s_{BA}}{v_{BA}} = \frac{150}{50} = 3 \text{ s}$



Ex. Two cars C_1 and C_2 moving in the same direction on a straight single lane road with velocities 12 m/s and 10 m/s respectively. When the separation between the two was 200 m C_2 started accelerating to avoid collision. What is the minimum acceleration of car C_2 so that they don't collide.



Sol. : Acceleration of car 1 w.r.t. car 2

$$\vec{a}_{12} = \vec{a}_1 - \vec{a}_2 = \vec{a}_{C_1} - \vec{a}_{C_2} = 0 - a = (-a)$$

$$\vec{u}_{12} = \vec{u}_1 - \vec{u}_2 = 12 - 10 = 2 \text{ m/s.}$$

The collision is just avoided if relative velocity becomes zero just at the moment the two cars meet each other.

i.e. $v_{12} = 0$ When $s_{12} = 200$

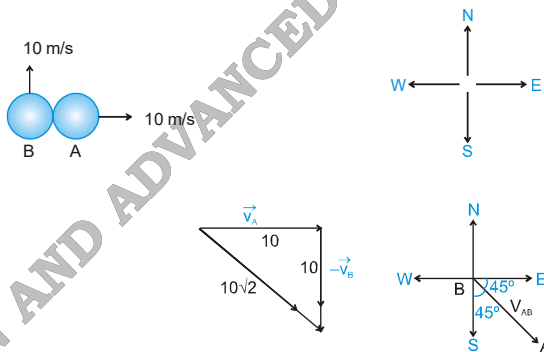
Now $v_{12} = 0$, $\vec{u}_{12} = 2$, $\vec{a}_{12} = -a$ and $s_{12} = 200$

$$\therefore v_{12}^2 - u_{12}^2 = 2a_{12}s_{12}$$

$$\Rightarrow 0 - 2^2 = -2 \times a \times 200 \Rightarrow a = \frac{1}{100} \text{ m/s}^2 = 0.1 \text{ m/s}^2 = 1 \text{ cm/s}^2.$$

$$\therefore \text{Minimum acceleration needed by car } C_2 = 1 \text{ cm/s}^2$$

Ex. Object A and B both have speed of 10 m/s. A is moving towards East while B is moving towards North starting from the same point as shown. Find velocity of A relative to B (\vec{v}_{AB})



Sol. Method 1

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

$$\Rightarrow |\vec{v}_{AB}| = 10\sqrt{2}$$

Method 2

$$\vec{v}_A = 10\hat{i}, \vec{v}_B = 10\hat{j}$$

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 10\hat{i} - 10\hat{j} \quad \therefore |\vec{v}_{AB}| = 10\sqrt{2}$$

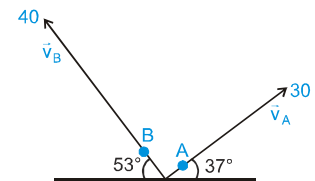
Note : $|\vec{v}_A - \vec{v}_B| = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos \theta}$, where θ is angle between \vec{v}_A and \vec{v}_B

Ex. Two particles A and B are projected in air. A is thrown with a speed of 30 m/sec and B with a speed of 40 m/sec as shown in the figure. What is the separation between them after 1 sec.

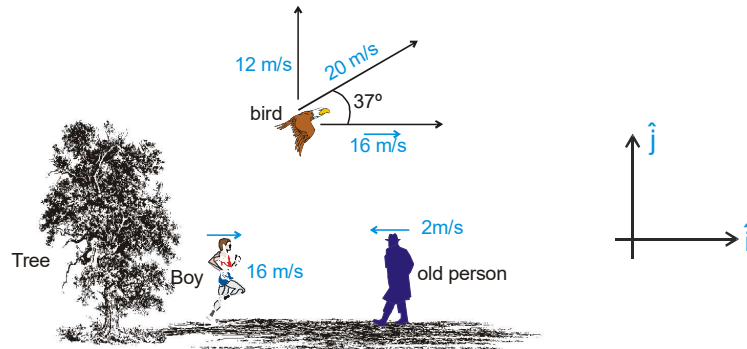
Sol. $\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = \vec{g} - \vec{g} = 0$

$$\therefore \vec{v}_{AB} = \sqrt{30^2 + 40^2} = 50$$

$$\therefore s_{AB} = v_{AB}t = 50t = 50 \text{ m}$$



Ex. An old man and a boy are walking towards each other and a bird is flying over them as shown in the figure.



- (1) Find the velocity of tree, bird and old man as seen by boy.
- (2) Find the velocity of tree, bird and boy as seen by old man
- (3) Find the velocity of tree, boy and old man as seen by bird.

Sol. (1) With respect to boy :

$$v_{\text{tree}} = 16 \text{ m/s } (\leftarrow) \text{ or } -16 \hat{i} \quad \Rightarrow \quad v_{\text{bird}} = 12 \text{ m/s } (\uparrow) \text{ or } 12 \hat{j}$$

$$v_{\text{old man}} = 18 \text{ m/s } (\leftarrow) \quad \text{or} \quad -18 \hat{i}$$

(2) With respect to old man :

$$v_{\text{Boy}} = 18 \text{ m/s } (\rightarrow) \quad \text{or} \quad 18 \hat{i}$$

$$v_{\text{Tree}} = 2 \text{ m/s } (\rightarrow) \quad \text{or} \quad 2 \hat{i}$$

$$v_{\text{Bird}} = 18 \text{ m/s } (\rightarrow) \text{ and } 12 \text{ m/s } (\uparrow) \quad \text{or} \quad 18 \hat{i} + 12 \hat{j}$$

(3) With respect to Bird :

$$v_{\text{Tree}} = 12 \text{ m/s } (\downarrow) \text{ and } 16 \text{ m/s } (\leftarrow) \quad \text{or} \quad -12 \hat{j} - 16 \hat{i}$$

$$v_{\text{old man}} = 18 \text{ m/s } (\leftarrow) \quad \text{and} \quad 12 \text{ m/s } (\downarrow) \quad \text{or} \quad -18 \hat{i} - 12 \hat{j}$$

$$v_{\text{Boy}} = 12 \text{ m/s } (\downarrow). \quad \text{or} \quad -12 \hat{j}$$

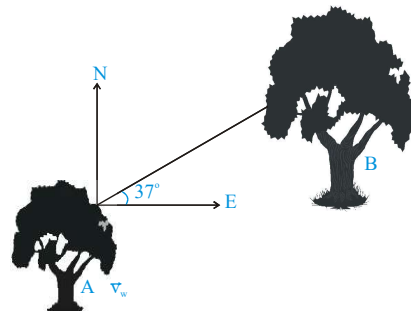
Question Based on Relative Motion are usually of following four types:

(a) **Minimum distance between two bodies in motion:**

When two bodies are in motion, the question like, the minimum distance between them or the time. When one body overtakes the other can be solved easily by the principle of relative motion. In these type of problems. One body is assumed to be at rest and the relative motion of the other body is considered. By assuming so two body problem is converted into one body problem and the solution becomes easy following example will illustrate the statement.

Ex. Wind is blowing in the east direction with a speed of 2 m/s. A bird wishes to travel from tree A to tree B. Tree B is 100m away from A in a direction 37° north of east the velocity of bird in still air is 4 m/s.

- (a) Find the direction in which bird should fly so that it can reach from A to B directly.
- (b) Find the actual velocity of the bird during the flight ?
- (c) Find the time taken by the bird to reach B ?



Sol.

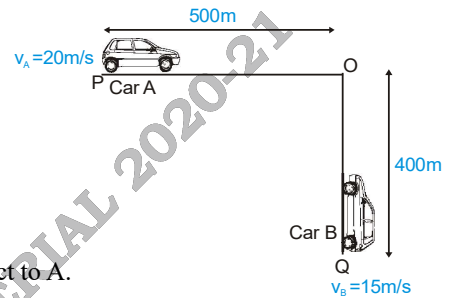
(a) $4 \sin \alpha = 2 \sin 37^\circ \Rightarrow \alpha \sin^{-1} \left(\frac{3}{10} \right)$

$\Rightarrow 37^\circ \sin^{-1} \left(\frac{3}{10} \right)$ with east

(b) $\vec{v}_b = \vec{v}_{bw} + \vec{v}_w$
 $= \vec{v}_w \cos 37^\circ + 4 \cos \alpha$
 $= 2 \times \frac{4}{5} + 4 \times \frac{\sqrt{91}}{10} = \frac{8 + 2\sqrt{91}}{5}$

(c) $t = \frac{100 \times 5}{8 + 2\sqrt{91}} = \frac{250}{4 + \sqrt{91}}$ sec

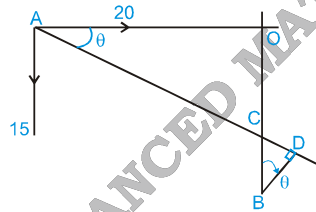
Ex. Two roads intersect at right angles. Car A is situated at P which is 500m from the intersection O on one of the roads. Car B is situated at Q which is 400m from the intersection on the other road. They start out at the same time and travel towards the intersection at 20m/s and 15m/s respectively. What is the minimum distance between them? How long do they take to reach it?



Sol. First we find out the velocity of car A relative to B.

As can be seen from (fig.), the magnitude of velocity of B with respect to A.

$v_A = 20 \text{ m/s}, v_B = 15 \text{ m/s}, OP = 500 \text{ m}; OQ = 400 \text{ m}$



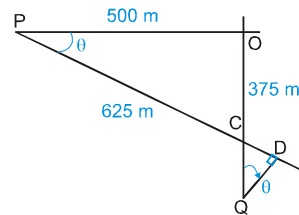
$\tan \theta = \frac{15}{20} = \frac{3}{4}; \cos \theta = \frac{4}{5}; \sin \theta = \frac{3}{5}$

$OC = AO \tan \theta = 500 \times \frac{3}{4} = 375 \text{ m}$

$BC = OB - OC = 400 - 375 = 25 \text{ m}$

$BD = BC(\cos \theta) = 25 \times \frac{4}{5} = 20 \text{ m}$

shortest distance = 20 m
 $PD = PC + CD = 625 + 15 = 640$



$|\vec{v}_{AB}| = 25 \text{ m/s} \Rightarrow t = \frac{640}{25} = 25.6 \text{ sec}$

(b) **Relative motion in Lift:**

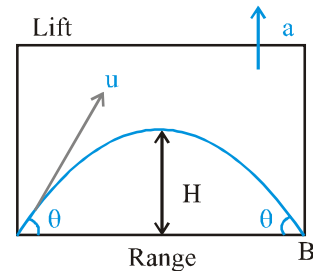
Projectile motion in a lift moving with acceleration a upward

1. In the reference frame of lift, acceleration of a freely falling object is $g + a$
2. Velocity at maximum height = $u \cos \theta$.

3. $T = \frac{2u \sin \theta}{g + a}$

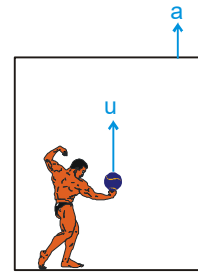
4. Maximum height (H) = $\frac{u^2 \sin^2 \theta}{2(g + a)}$

5. Range = $\frac{u^2 \sin 2\theta}{g + a}$



Ex. A lift is moving up with acceleration a . A person inside the lift throws the ball upwards with a velocity u relative to hand.

- (a) What is the time of flight of the ball?
 (b) What is the maximum height reached by the ball in the lift?



Sol. (a) $\vec{a}_{BL} = \vec{a}_B - \vec{a}_L = g + a \Rightarrow \vec{s} = \vec{u}t + \frac{1}{2} \vec{a}_{BL} t^2$
 $\Rightarrow 0 = uT - \frac{1}{2}(g+a)T^2 \therefore T = \frac{2u}{(g+a)}$
 (b) $v^2 - u^2 = 2as \Rightarrow 0 - u^2 = -2(g+a)H$
 $H = \frac{u^2}{2(g+a)}$

(c) Relative motion in River Flow:

If a man can swim relative to water with velocity \vec{V}_{MR} and water is flowing relative to ground with velocity \vec{V}_R , velocity of man relative to ground \vec{V}_M will be given by.

$$\vec{V}_{MR} = \vec{V}_M - \vec{V}_R \quad \text{or} \quad \vec{V}_M = \vec{V}_{MR} + \vec{V}_R$$

if $\vec{V}_R = 0$, then $\vec{V}_M = \vec{V}_{MR}$

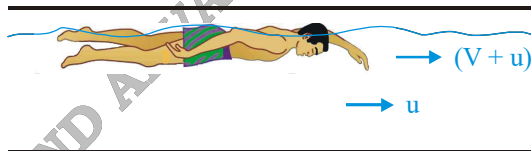
In words, velocity of man in still water = velocity of man w.r.t. river.

River problems in one Dimension:

Velocity of river is u & velocity of man in still water is v

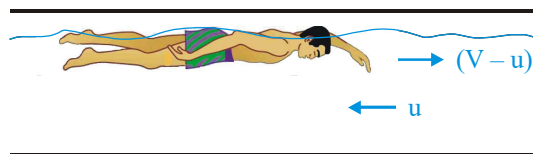
Case-I Man swimming downstream (along the direction of river flow). In this case velocity of river $V_R = +u$. Velocity of man w.r.t. river $V_{MR} = +V$

now $\vec{V}_M = \vec{V}_{MR} + \vec{V}_R = (u + V)$



Case-II Man swimming upstream (opposite to the direction of river flow). In this case velocity of river $\vec{V}_R = -u$ velocity of man w.r.t. river = $+v$

now $\vec{V}_M = \vec{V}_{MR} + \vec{V}_R = (V - u)$



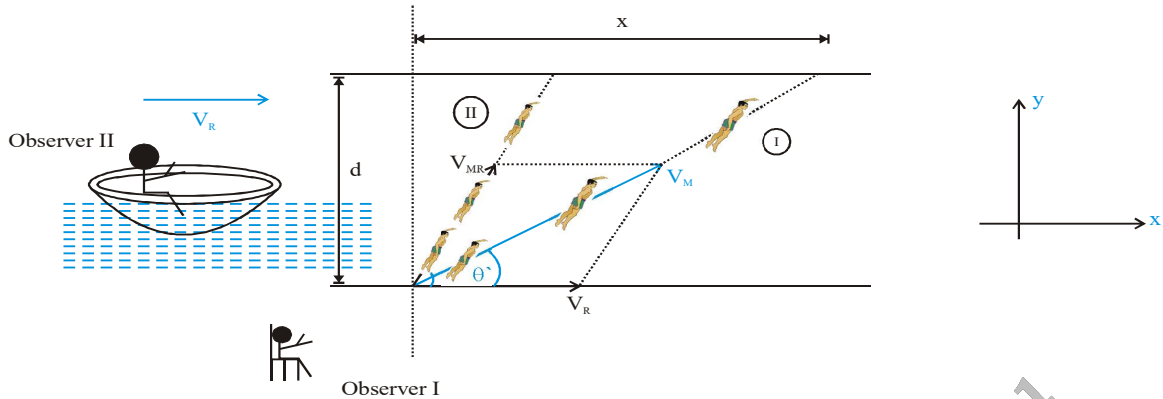
Motion of man swimming in a river:

Consider a man swimming in a river with a velocity of \vec{V}_{MR} relative to river at an angle of θ with the river flow. The velocity of river is \vec{V}_R

Let there be two observer I and II, observer I is on ground and observer II is on a raft moving with the same velocity as that of river. Hence motion w.r.t. river. Hence motion w.r.t. observer II is same as motion w.r.t. river, i.e., the man will appear to swim at an angle θ with the river flow for observer II for observer I the velocity of swimmer will be

$$\vec{V}_M = \vec{V}_{MR} + \vec{V}_R$$

Hence the, swimmer will appear to move at an angle θ with the river flow



- (I) : Motion of swimmer for observer I.
- (II) : Motion of swimmer for observer II.

River problem in two dimension (crossing river) :-

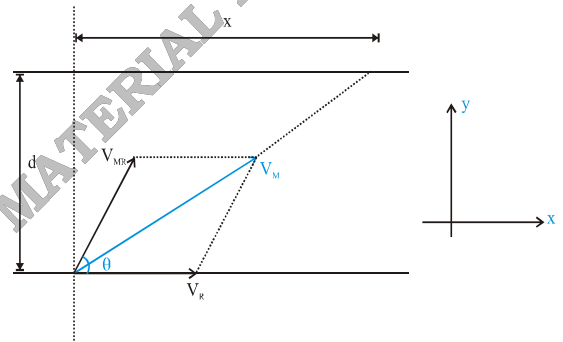
Consider a man swimming in a river with a velocity of \vec{V}_{MR} relative to river at an angle of θ with the river flow.

The velocity of river is V_R and width of the river is d

$$\vec{V}_M = \vec{V}_{MR} + \vec{V}_R$$

$$\Rightarrow \vec{V}_M = (V_{MR} \cos \theta \mathbf{i} + V_{MR} \sin \theta \mathbf{j}) + V_R \mathbf{i}$$

$$\Rightarrow \vec{V}_M = (V_{MR} \cos \theta + V_R) \mathbf{i} + V_{MR} \sin \theta \mathbf{j}$$



Here $V_{MR} \sin \theta$ is the component of velocity of man in the direction perpendicular to the river flow. This component of velocity is responsible for the man crossing the river. Hence if the time to cross the river is t , then

$$t = \frac{d}{v_y} = \frac{d}{V_{MR} \sin \theta}$$

DRIFT

It is defined as the displacement of man in the direction of river flow. (see the figure).

It is simply the displacement along x-axis, during the period the man crosses the river. $(v_{MR} \cos \theta + v_R)$ is the component of velocity of man in the direction of river flow and this component of velocity is responsible for drift along the river flow. If drift is x then,

$$\text{Drift} = v_x \times t$$

$$x = (v_{MR} \cos \theta + v_R) \times \frac{d}{V_{MR} \sin \theta}$$

Crossing the river in shortest time

As we know that $t = \frac{d}{V_{MR} \sin \theta}$. Clearly t will be minimum when $\theta = 90^\circ$ i.e. time to cross the river will be minimum

if man swims perpendicular to the river flow. Which is equal to $\frac{d}{V_{MR}}$.

Crossing the river in shortest path, Minimum Drift

The minimum possible drift is zero. In this case the man swims in the direction perpendicular to the river flow as seen from the ground. This path is known as **shortest path**

here $x_{\min} = 0 \Rightarrow (v_{MR} \cos\theta + v_R) = 0$

or $\cos\theta = -\frac{v_R}{v_{MR}}$

☞ since $\cos\theta$ is $-ve$, $\therefore \theta > 90^\circ$, i.e. for minimum drift the man must swim at some angle ϕ with the perpendicular in backward direction.

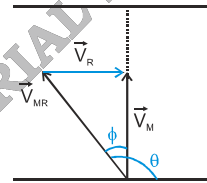
Where $\sin\phi = \frac{v_R}{v_{MR}}$

☞ $\theta = \cos^{-1}\left(\frac{-v_R}{v_{MR}}\right) \therefore \left|\frac{v_R}{v_{MR}}\right| \leq 1$ i.e. $v_R \leq v_{MR}$

i.e. minimum drift is zero if and only if velocity of man in still water is greater than or equal to the velocity of river.

☞ Time to cross the river along the shortest path

$$t = \frac{d}{v_{MR} \sin\theta} = \frac{d}{\sqrt{v_{MR}^2 - v_R^2}}$$



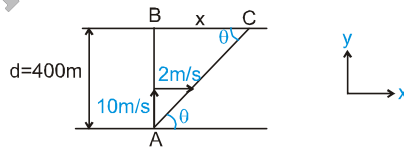
Note: If $v_R > v_{MR}$ then it is not possible to have zero drift. In this case the minimum drift (corresponding to shortest possible path) is non zero and the condition for minimum drift can be proved to be $\cos\theta = -\frac{v_{MR}}{v_R}$ or

$\sin\phi = \frac{v_{MR}}{v_R}$ for minimum but non zero drift.

Ex. A 400 m wide river is flowing at a rate of 2.0 m/s. A boat is sailing with a velocity of 10 m/s with respect to the water, in a direction perpendicular to the river.

- (a) Find the time taken by the boat to reach the opposite bank.
- (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank.
- (c) In what direction does the boat actually move, with river flow (downstream).

Sol.



(a) time taken to cross the river

$$t = \frac{d}{v_y} = \frac{400\text{m}}{10\text{m/s}} = 40\text{ s} \quad \text{Ans.}$$

(b) drift $(x) = (v_x)(t) = (2\text{m/s})(40\text{s}) = 80\text{ m} \quad \text{Ans.}$

(c) Actual direction of boat,

$$\theta = \tan^{-1}\left(\frac{10}{2}\right) = \tan^{-1} 5, \text{ (downstream) with the river flow.}$$

Ex. A man can swim at the rate of 5 km/h in still water. A 1 km wide river flows at the rate of 3 km/h. The man wishes to swim across the river directly opposite to the starting point.

- (a) Along what direction must the man swim?
- (b) What should be his resultant velocity?
- (c) How much time will he take to cross the river?

Sol. The velocity of man with respect to river $v_{mR} = 5$ km/hr, this is greater than the river flow velocity, therefore, he can cross the river directly (along the shortest path). The angle of swim must be

$$\theta = \frac{\pi}{2} + \sin^{-1} \left(\frac{v_r}{v_{mR}} \right) = 90^\circ + \sin^{-1} \left(\frac{3}{5} \right) = 90^\circ + \sin^{-1} \left(\frac{3}{5} \right) = 90^\circ + 37^\circ$$

= 127° w.r.t. the river flow or 37° w.r.t. perpendicular in backward direction

Ans.

(b) Resultant velocity will be $v_m = \sqrt{v_{mR}^2 - v_R^2} = \sqrt{5^2 - 3^2} = 4$ km/hr

along the direction perpendicular to the river flow.

(c) time taken to cross the

$$t = \frac{d}{\sqrt{v_{mR}^2 - v_R^2}} = \frac{1 \text{ km}}{4 \text{ km/hr}} = \frac{1}{4} \text{ h} = 15 \text{ min}$$

Ex. A man wishing to cross a river flowing with velocity u jumps at an angle θ with the river flow.

- (i) Find the net velocity of the man with respect to ground if he can swim with speed v in still water.
- (ii) In what direction does the man actually move.
- (iii) Find how far from the point directly opposite to the starting point does the man reach the opposite bank, if the width of the river is d . (i.e. drift)

Sol. : (i) $v_{MR} = v, v_R = u$

$$\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$$

∴ Velocity of man, $v_M = \sqrt{u^2 + v^2 + 2vu \cos \theta}$ **Ans.**

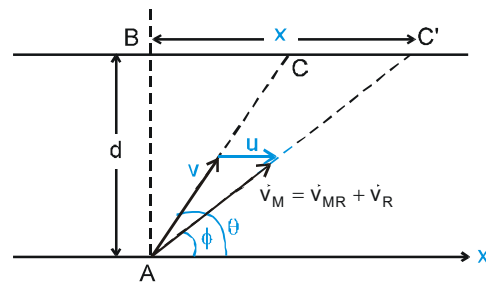
(ii) $\tan \phi = \frac{v \sin \theta}{u + v \cos \theta}$ **Ans.**

(iii) $(v \sin \theta) t = d$

$$\Rightarrow t = \frac{d}{v \sin \theta}$$

$$x = (u + v \cos \theta) t$$

$$= (u + v \cos \theta) \frac{d}{v \sin \theta} \quad \text{Ans.}$$



Ex. A boat moves relative to water with a velocity v which is n times less than the river flow velocity u . At what angle to the stream direction must the boat move to minimize drifting?

Sol. (In this problem, one thing should be carefully noted that the velocity of boat is less than the river flow velocity. Hence boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero.)

Suppose boat starts at an angle θ from the normal direction up stream as shown.

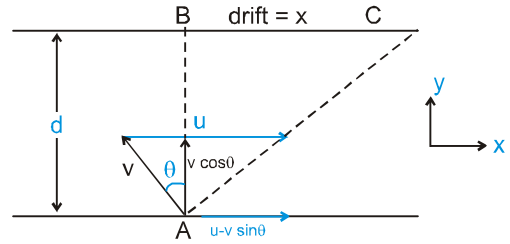
Component of velocity of boat along the river, $v_x = u - v \sin \theta$
and velocity perpendicular to the river, $v_y = v \cos \theta$.

$$\text{time taken to cross the river is } t = \frac{d}{v_y} = \frac{d}{v \cos \theta}.$$

$$\text{Drift } x = (v_x)t$$

$$= (u - v \sin \theta) \frac{d}{v \cos \theta}$$

$$= \frac{ud}{v} \sec \theta - d \tan \theta$$



The drift x is minimum, when $\frac{dx}{d\theta} = 0$,

$$\text{or } \left(\frac{ud}{v}\right) (\sec \theta \cdot \tan \theta) - d \sec^2 \theta = 0$$

$$\text{or } \frac{u}{v} \sin \theta = 1 \quad \text{or } \sin \theta = \frac{v}{u}$$

This is the result we stated without proof as a note in section 4.5

so, for minimum drift, the boat must move at an angle $\theta = \sin^{-1}\left(\frac{v}{u}\right) = \sin^{-1} \frac{1}{n}$ from normal direction.

(d) Wind Airplane Problems

This is very similar to boat river flow problems. The only difference is that boat is replaced by aeroplane and river is replaced by wind.

Thus,

velocity of aeroplane with respect to wind

$$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w$$

$$\text{or } \vec{v}_a = \vec{v}_{aw} + \vec{v}_w$$

where, \vec{v}_a = velocity of aeroplane w.r.t. ground

and, \vec{v}_w = velocity of wind.

Ex. An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is ℓ and the aeroplane maintains the constant speed v w.r.t. wind. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.

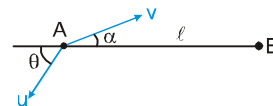
Sol. Suppose plane is oriented at an angle α w.r.t. line AB while the plane is moving from A to B :

$$\text{Velocity of plane along AB} = v \cos \alpha - u \cos \theta,$$

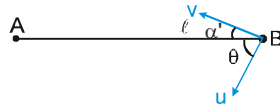
and for no-drift from line AB

$$v \sin \alpha = u \sin \theta \quad \Rightarrow \quad \sin \alpha = \frac{u \sin \theta}{v}$$

$$\text{time taken from A to B : } t_{AB} = \frac{\ell}{v \cos \alpha - u \cos \theta}$$



Suppose plane is oriented at an angle α' w.r.t. line AB while the plane is moving from B to A :



velocity of plane along BA = $v \cos \alpha + u \cos \theta$ and for no drift from line AB
 $v \sin \alpha = u \sin \theta$

$$\Rightarrow \sin \alpha = \frac{u \sin \theta}{v}$$

$$\Rightarrow \alpha = \alpha'$$

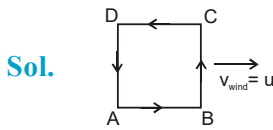
time taken from B to A : $t_{BA} = \frac{\ell}{v \cos \alpha + u \cos \theta}$

total time taken = $t_{AB} + t_{BA} = \frac{\ell}{v \cos \alpha - u \cos \theta} + \frac{\ell}{v \cos \alpha + u \cos \theta}$

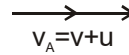
$$= \frac{2v\ell \cos \alpha}{v^2 \cos^2 \alpha - u^2 \cos^2 \theta} \Rightarrow \frac{2v\ell \sqrt{1 - \frac{u^2 \sin^2 \theta}{v^2}}}{v^2 - u^2}$$

Ex. Find the time an aeroplane having velocity v , takes to fly around a square with side a if the wind is blowing at a velocity u along one side of the square.

Ans. $\frac{2a}{v^2 - u^2} \left(v + \sqrt{v^2 - u^2} \right)$

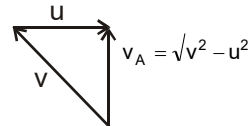


Velocity of aeroplane while flying through AB



$$v_A = v + u \Rightarrow t_{AB} = \frac{a}{v + u}$$

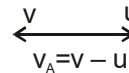
Velocity of aeroplane while flying through BC



$$v_A = \sqrt{v^2 - u^2}$$

$$t_{BC} = \frac{a}{\sqrt{v^2 - u^2}}$$

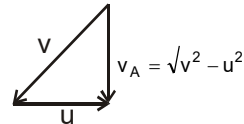
Velocity of aeroplane while flying through CD



$$v_A = v - u$$

$$t_{CD} = \frac{a}{v - u}$$

Velocity of aeroplane while flying through DA



$$v_A = \sqrt{v^2 - u^2}$$

$$t_{DA} = \frac{a}{\sqrt{v^2 - u^2}}$$

Total time = $t_{AB} + t_{BC} + t_{CD} + t_{DA}$

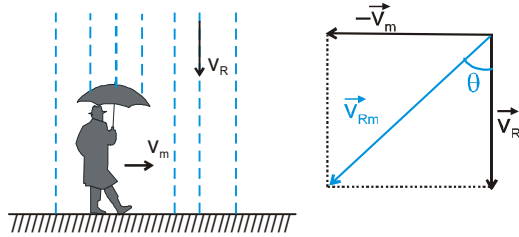
$$= \frac{a}{v + u} + \frac{a}{\sqrt{v^2 - u^2}} + \frac{a}{v - u} + \frac{a}{\sqrt{v^2 - u^2}} = \frac{2a}{v^2 - u^2} \left(v + \sqrt{v^2 - u^2} \right)$$

(e) Rain Problem

If rain is falling vertically with a velocity \vec{v}_R and an observer is moving horizontally with velocity \vec{v}_m , the velocity of rain relative to observer will be :

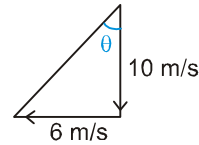
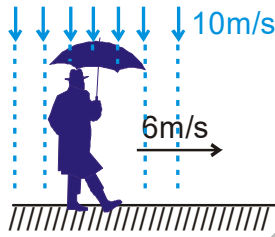
$$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_m \quad \text{or} \quad v_{Rm} = \sqrt{v_R^2 + v_m^2}$$

and direction $\theta = \tan^{-1} \left(\frac{v_m}{v_R} \right)$ with the vertical as shown in figure.



Ex. Rain is falling vertically at speed of 10 m/s and a man is moving with velocity 6 m/s. Find the angle at which the man should hold his umbrella to avoid getting wet.

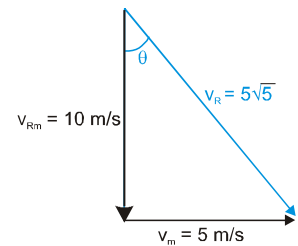
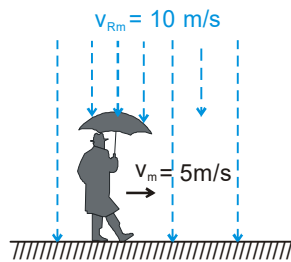
Sol. :



$$\begin{aligned} \vec{v}_{\text{rain}} &= -10 \hat{j} \quad \Rightarrow \quad \vec{v}_{\text{man}} = 6 \hat{j} \\ \vec{v}_{\text{r.w.r.t. man}} &= -10 \hat{j} - 6 \hat{i} \\ \tan \theta &= \frac{6}{10} \quad \Rightarrow \quad \theta = \tan^{-1} \left(\frac{3}{5} \right) \end{aligned}$$

Where θ is angle with vertical

Ex. A man moving with 5m/s observes rain falling vertically at the rate of 10 m/s. Find the speed and direction of the rain with respect to ground.



Sol. : $v_{RM} = 10 \text{ m/s}, v_M = 5 \text{ m/s}$

$$\vec{v}_{RM} = \vec{v}_R - \vec{v}_M$$

$$\Rightarrow \vec{v}_R = \vec{v}_{RM} + \vec{v}_M \Rightarrow v_R = 5\sqrt{5}$$

$$\tan \theta = \frac{1}{2}, \theta = \tan^{-1} \frac{1}{2}$$

- Ex.** A standing man, observes rain falling with velocity of 20 m/s at an angle of 30° with the vertical.
- (i) Find the velocity with which the man should move so that rain appears to fall vertically to him.
- (ii) Now if he further increases his speed, rain again appears to fall at 30° with the vertical. Find his new velocity.

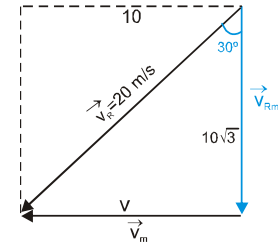
Sol. (i) $\vec{v}_m = -v \hat{i}$ (let)

$$\vec{v}_R = -10 \hat{i} - 10\sqrt{3} \hat{j}$$

$$\vec{v}_{RM} = -(10-v) \hat{i} - 10\sqrt{3} \hat{j}$$

$$\Rightarrow -(10-v) = 0 \text{ (for vertical fall, horizontal component must be zero)}$$

or $v = 10 \text{ m/s}$ **Ans.**



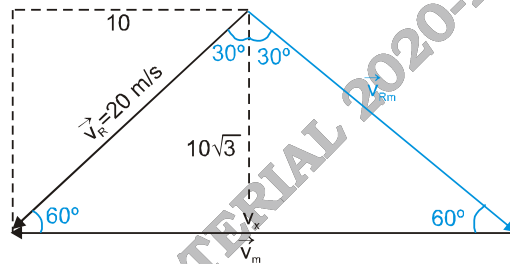
(ii) $\vec{v}_R = -10 \hat{i} - 10\sqrt{3} \hat{j}$

$$\vec{v}_m = -v_x \hat{i}$$

$$\vec{v}_{RM} = -(10-v_x) \hat{i} - 10\sqrt{3} \hat{j}$$

Angle with the vertical = 30°

$$\Rightarrow \tan 30^\circ = \frac{10-v_x}{-10\sqrt{3}} \Rightarrow v_x = 20 \text{ m/s}$$

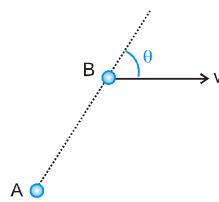


VELOCITY OF APPROACH / SEPARATION IN TWO DIMENSION

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

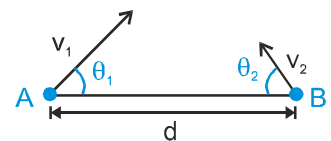
- Ex.** Particle A is at rest and particle B is moving with constant velocity v as shown in the diagram at $t = 0$. Find their velocity of separation



Sol. $v_{BA} = v_B - v_A = v$

$$v_{sep} = \text{component of } v_{BA} \text{ along line AB} = v \cos \theta$$

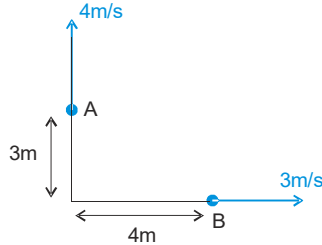
- Ex.** Two particles A and B are moving with constant velocities v_1 and v_2 . At $t = 0$, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B. Find their velocity of approach.



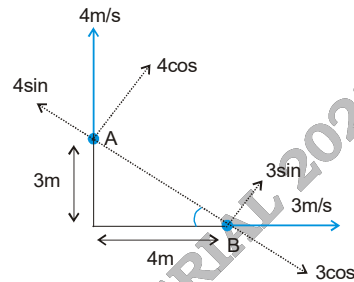
- Sol.** Velocity of approach is relative velocity along line AB

$$v_{APP} = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

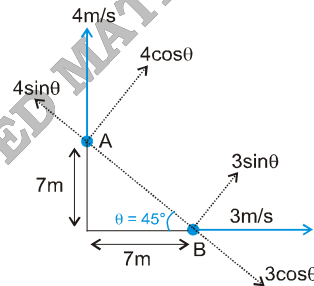
- Ex.** Particles A and B are moving as shown in the diagram at $t = 0$. Find their velocity of separation
- at $t = 0$
 - at $t = 1$ sec.



- Sol.** (i) $\tan \theta = 3/4$
 v_{sep} = relative velocity along line AB
 $= 3\cos\theta + 4\sin\theta$
 $= 3 \cdot \frac{4}{5} + 4 \cdot \frac{3}{5} = \frac{24}{5} = 4.8 \text{ m/s}$



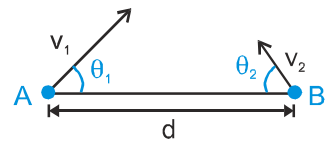
- (ii) $\theta = 45^\circ$
 v_{sep} = relative velocity along line AB
 $= 3\cos\theta + 4\sin\theta$
 $= 3 \cdot \frac{1}{\sqrt{2}} + 4 \cdot \frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}} \text{ m/s}$



Condition for uniformly moving particles to collide

If two particles are moving with uniform velocities and the relative velocity of one particle w.r.t. other particle is directed towards each other then they will collide.

- Ex.** Two particles A and B are moving with constant velocities v_1 and v_2 . At $t = 0$, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B.



- Find the condition for A and B to collide.
 - Find the time after which A and B will collide if separation between them is d at $t = 0$
- Sol.** (i) For A and B to collide, their relative velocity must be directed along the line joining them. Therefore their relative velocity along the perpendicular to this line must be zero.

Thus $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

- (ii) $v_{\text{APP}} = v_1 \cos\theta_1 + v_2 \cos\theta_2$

$$t = \frac{d}{v_{\text{app}}} = \frac{d}{v_1 \cos\theta_1 + v_2 \cos\theta_2}$$

Minimum / Maximum distance between two particles

If the separation between two particles decreases and after some time it starts increasing then the separation between them will be minimum at the instant, velocity of approach changes to velocity of separation. (at this instant $v_{app} = 0$)

Mathematically S_{AB} is minimum when $\frac{dS_{AB}}{dt} = 0$

Similarly for maximum separation $v_{sep} = 0$.

Note : If the initial position of two particles are \vec{r}_1 and \vec{r}_2 and their velocities are \vec{v}_1 and \vec{v}_2 then shortest distance

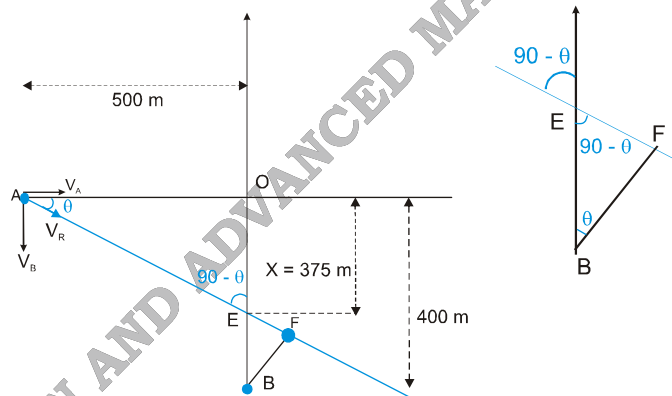
between the particles, $d_{shortest} = \frac{|\vec{r}_{12} \times \vec{v}_{12}|}{|\vec{v}_{12}|}$ and time after which this situation will occur, $t = -\frac{\vec{r}_{12} \cdot \vec{v}_{12}}{|\vec{v}_{12}|^2}$

Ex. Two cars A and B are moving west to east and south to north respectively along crossroads. A moves with a speed of 72 kmh^{-1} and is 500 m away from point of intersection of cross roads and B moves with a speed of 54 kmh^{-1} and is 400 m away from point of intersection of cross roads. Find the shortest distance between them?

Sol. **Method – I (Using the concept of relative velocity)**

In this method we watch the velocity of A w.r.t. B. To do this we plot the resultant velocity V_r . Since the accelerations of both the bodies is zero, so the relative acceleration between them is also zero. Hence the relative velocity will remain constant. So the path of A with respect to B will be straight line and along the direction of relative velocity of A with respect to B. The shortest distance between A & B is when A is at point F (i.e. when we drop a perpendicular from B on the line of motion of A with respect to B).

From figure



$$\tan\theta = \frac{V_B}{V_A} = \frac{15}{20} = \frac{3}{4} \dots\dots(i)$$

This θ is the angle made by the resultant velocity vector with the x-axis.

Also we know that from figure

$$OE = \frac{x}{500} = \frac{3}{4} \dots\dots(ii)$$

From equation (i) & (ii) we get

$$x = 375 \text{ m}$$

$$\therefore EB = OB - OE = 400 - 375 = 25 \text{ m}$$

But the shortest distance is BF.

$$\text{From magnified figure we see that } BF = EB \cos\theta = 25 \times \frac{4}{5}$$

$$\therefore BF = 20 \text{ m}$$

Method II

(Using the concept of maxima – minima)

A & B be are the initial positions and A', B' be the final positions after time t.

B is moving with a speed of 15 m/sec so it will travel a distance of $BB' = 15t$ during time t.

A is moving with a speed of 20 m/sec so it will travel a distance of $AA' = 20t$ during time t.

So

$$OA' = 500 - 20t$$

$$OB' = 400 - 15t$$

$$\therefore A'B'^2 = OA'^2 + OB'^2 = (500 - 20t)^2 + (400 - 15t)^2 \dots (i)$$

For A'B' to be minimum $A'B'^2$ should also be minimum

$$\therefore \frac{d(A'B'^2)}{dt} = \frac{d(400 - 15t)^2 + (500 - 20t)^2}{dt} = 0$$

$$= 2(400 - 15t)(-15) + 2(500 - 20t)(-20) = 0 = -1200 + 45t = 2000 - 80t$$

$$\therefore 125t = 3200$$

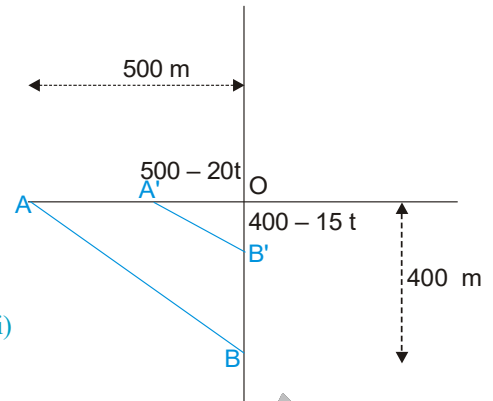
$$\therefore t = \frac{128}{5} \text{ s. Hence A and B will be closest after } \frac{128}{5} \text{ s.}$$

Now $\frac{d^2 A'B'}{dt^2}$ comes out to be positive hence it is a minima.

On substituting the value of t in equation (i) we get

$$\therefore A'B'^2 = \left(400 - 15 \times \frac{128}{5}\right)^2 + \left(500 - 20 \cdot \frac{128}{5}\right)^2 = \sqrt{16^2 + (-12)^2} = 20 \text{ m}$$

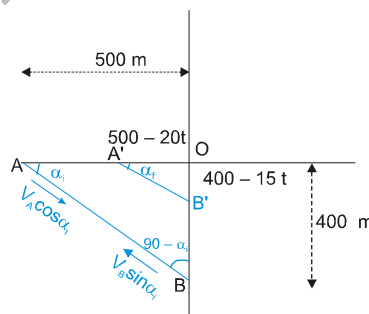
$$\therefore \text{Minimum distance } A'B' = 20 \text{ m.}$$



Method III (Using the concept of relative velocity of approach)

After time t let us plot the components of velocity of A and B in the direction along AB. When the distance between the two is minimum, the relative velocity of approach is zero.

$$\therefore V_A \cos \alpha_f + V_B \sin \alpha_f = 0$$



(where α_f is the angle made by the line A'B' with the x-axis)

$$20 \cos \alpha_f = -15 \sin \alpha_f$$

$$\therefore \tan \alpha_f = -\frac{20}{15} = -\frac{4}{3}$$

(Here do not confuse this angle with the angle θ in method (I) because that θ is the angle made by the net relative velocity with x-axis, but α_f is the angle made by the line joining the two particles with x-axis when velocity of approach is zero.)

$$\therefore \frac{400 - 15t}{500 - 20t} = -\frac{4}{3}$$

$$\therefore t = \frac{128}{5} \quad \text{So, } OB' = 16 \text{ m and } OA' = -12 \text{ m}$$

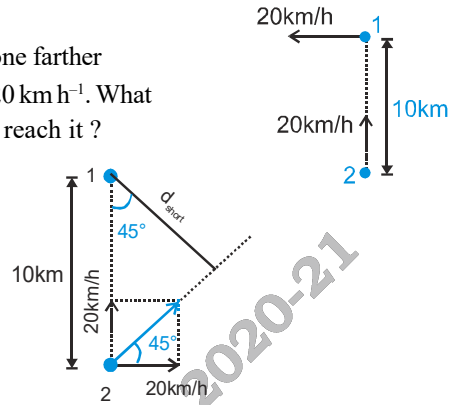
$$A'B' = \sqrt{16^2 + (-12)^2} = 20 \text{ m}$$

Ex. Two ships are 10 km apart on a line joining south to north. The one farther north is steaming west at 20 km h⁻¹. The other is steaming north at 20 km h⁻¹. What is their distance of closest approach? How long do they take to reach it?

Sol. Solving from the frame of particle -1

$$\text{we get } d_{\text{short}} = 10 \cos 45^\circ = \frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ km}$$

$$t = \frac{10 \sin 45^\circ}{|\vec{V}_{21}|} = \frac{10 \times 1/\sqrt{2}}{20\sqrt{2}} = \frac{1}{4} \text{ h} = 15 \text{ min.}$$



Ex. Two particles A and B are moving with uniform velocity as shown in the figure given below at t = 0.

- (i) Will the two particles collide
- (ii) Find out shortest distance between two particles

Sol. Solving from the frame of B

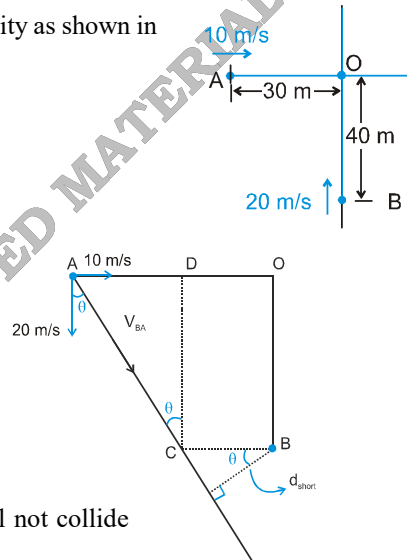
$$\text{we get } \tan \theta = \frac{10}{20} = \frac{1}{2}$$

$$\text{again } \tan \theta = \frac{AD}{CD} = \frac{AD}{40} = \frac{1}{2}$$

$$\Rightarrow AD = 20 \Rightarrow DO = 10 \Rightarrow BC = 10$$

$$d_{\text{short}} = BC \cos \theta = 10 \cos \theta = \frac{10 \times 2}{\sqrt{5}} = 4\sqrt{5} \text{ m}$$

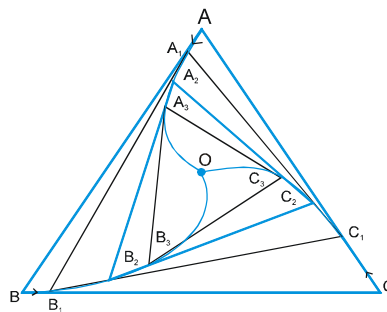
Since closest distance is non zero therefore they will not collide



Miscellaneous Problems on collision

Ex. There are particles A, B and C are situated at the vertices of an equilateral triangle ABC of side a at t = 0. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particle meet each other?

Sol. The motion of the particles is roughly sketched in figure. By symmetry they will meet at the centroid O of the triangle. At any instant the particles will form an equilateral triangle ABC with the same



Sol. Let at any instant the velocity of B makes an angle α with that of x axis and the time to collide is T.

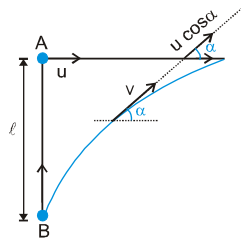
$$v_{\text{app}} = v - u \cos \alpha$$

$$\ell = \int_0^T v_{\text{app}} dt = \int_0^T (v - u \cos \alpha) dt \quad \dots\dots (1)$$

Now equating the displacement of A and B along x direction we get

$$uT = \int v \cos \alpha dt \quad \dots\dots (2)$$

Now from (1) and (2) we get



$$\ell = vT - \int_0^T u \cos \alpha dt = vT - \frac{u}{v} \int_0^T v \cos \alpha dt = vT - \frac{u}{v} \cdot uT \Rightarrow T = \frac{\ell v}{v^2 - u^2}$$

Now distance travelled by A and B

$$= u \times \frac{\ell v}{v^2 - u^2} \text{ and } v \times \frac{\ell v}{v^2 - u^2} = \frac{uv\ell}{v^2 - u^2} \text{ and } \frac{v^2 \ell}{v^2 - u^2}$$