SEQUENCE AND SERIES

SEQUENCE

A succession of terms a_1, a_2, a_3, a_4........ formed according to some rule or law.

Examples are : 1, 4, 9, 16, 25

–1, 1, –1, 1,........

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, ...$

REAL SEQUENCE

A sequence whose range is a subset of R is called a real sequence.

E.g. **(i)** 2, 5, 8, 11, ...
(ii) 4, 1, – 2, – 5, ...
(iii) 3, –9, 27, – 81, ...

A finite sequence has a finite (i.e. limited) number of terms. An infinite sequence has an unlimited number of terms, i.e. there is no last term.

SERIES

The indicated sum of the terms of a sequence. In the case of a finite sequence $a_1, a_2, a_3, ..., a_n$ the corresponding series is $a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

PROGRESSION

The word progression refers to sequence or series – finite or infinite

Arithmetic Progression (A.P.)

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as

$a, a + d, a + 2d, ..., a + (n – 1)\,d, ...$

(a) n^{th} term of AP $T_n = a + (n – 1)d$, where $d = t_n – t_{n–1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[a+\ell] = \frac{n}{2}[2a+(n-1)d]$

where l is the last term.

KEY POINTS

(i) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P. i.e. A.

(ii) Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e. 2A

(iii) Also n^{th} term $T_n = S_n – S_{n–1}$

Ex. If t_{54} of an A.P. is – 61 and t_4 = 64, find t_{10}.

Sol. Let a be the first term and d be the common difference

so $t_{54} = a + 53d = -61$ ....(i)

and $t_4 = a + 3d = 64$ ....(ii)

equation (i) – (ii) we get

$\Rightarrow$ $50d = -125$

$d = -\dfrac{5}{2}$

$\Rightarrow$ $a = \dfrac{143}{2}$ So $t_{10} = \dfrac{143}{2} + 9\left(-\dfrac{5}{2}\right) = 49$

Ex. If (x + 1), 3x and (4x + 2) are first three terms of an A.P. then find its 5th term.

Sol. (x + 1), 3x, (4x + 2) are in AP

$\Rightarrow$ $3x - (x+1) = (4x+2) - 3x$ $\Rightarrow$ $x = 3$

$\therefore$ $a = 4, d = 9 - 4 = 5$ $\Rightarrow$ $T_5 = 4 + 4(5) = 24$

Ex. Find the sum of all natural numbers divisible by 5, but less than 100.

Sol. All those numbers are 5, 10, 15, 20, ..., 95.

Here $a = 5,\ n = 19\ \&\ l = 95$

So $S = \dfrac{19}{2}(5 + 95) = 950.$

Ex. The sum of first n terms of two A.Ps. are in ratio $\dfrac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27} \Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11th terms

$\dfrac{n-1}{2} = 10$ $\Rightarrow$ $n = 21$

so ratio of 11th terms is $\dfrac{7(21)+1}{4(21)+27} = \dfrac{148}{111} = \dfrac{4}{3}$

Properties of A.P.

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow$ 2 b = a + c & if a, b, c, d are in A.P. $\Rightarrow$ a + d = b + c.

(iii) Three numbers in A.P. can be taken as a – d, a, a + d ;

four numbers in A.P. can be taken as a – 3d, a – d, a + d, a + 3d;

five numbers in A.P. are a – 2d, a – d, a, a + d, a + 2d ;

six terms in A.P. are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it.

$a_n = 1/2\,(a_{n-k} + a_{n+k}),\ k < n$. For k = 1, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$;

for $k = 2,\ a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

(vii) The sum and difference of two AP's is an AP.

(viii) k^{th} term from the last = $(n-k+1)^{th}$ term from the beginning.

Ex. Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then find the middle terms.

Sol. Let the numbers are $a-3d, a-d, a+d, a+3d$

given, $a-3d+a-d+a+d+a+3d=20 \Rightarrow 4a=20 \Rightarrow a=5$

and $(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120 \Rightarrow 4a^2+20d^2=120$

$\Rightarrow 4\times 5^2+20d^2=120$

$\Rightarrow d^2=1 \Rightarrow d=\pm 1$

Hence numbers are 2, 4, 6, 8

Ex. If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Sol. L.H.S. $=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$$=\frac{1}{\sqrt{a_2}+\sqrt{a_1}}+\frac{1}{\sqrt{a_3}+\sqrt{a_2}}+\ldots\ldots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$=\frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)}+\frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)}+\ldots\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let 'd' is the common difference of this A.P.

then $a_2-a_1=a_3-a_2=\ldots\ldots=a_n-a_{n-1}=d$

Now L.H.S.

$$=\frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_{n-2}}+\sqrt{a_n}-\sqrt{a_{n-1}}\right\}=\frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\}$$

$$=\frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}=\frac{a_1+(n-1)d-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}=\frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n}+\sqrt{a_1})}=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}=R.H.S.$$

Geometric Progression (G.P.)

G.P. is a sequence of non zero numbers each of the succeeding term is equal to the preceeding term multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term.

Therefore $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots$ is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term ; $T_n = a\,r^{n-1}$

(b) Sum of the first n terms; $S_n=\frac{a(r^n-1)}{r-1}$, if $r\neq 1$

(c) Sum of infinite G.P. , $S_\infty=\frac{a}{1-r}$; $0<|r|<1$

Ex. If the first term of G.P. is 7, its n^{th} term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Sol. Given a = 7

$t_n = ar^{n-1} = 7(r)^{n-1} = 448.$

$\Rightarrow \quad 7r^n = 448\, r$

Also $\quad S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{7(r^n-1)}{r-1}$

$\Rightarrow \quad 889 = \dfrac{448r-7}{r-1} \quad \Rightarrow \quad r = 2$

Hence $T_5 = ar^4 = 7(2)^4 = 112.$

Ex. Let $S = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ...$, find the sum of

(i) first 20 terms of the series **(ii)** infinite terms of the series.

Sol. **(i)** $S_{20} = \dfrac{\left(1-\left(\dfrac{1}{2}\right)^{20}\right)}{1-\dfrac{1}{2}} = \dfrac{2^{20}-1}{2^{19}}.$ **(ii)** $S_\infty = \dfrac{1}{1-\dfrac{1}{2}} = 2.$

Properties of G.P.

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a GP : a/r, a, ar ;

Four consecutive terms of a GP : a/r^3, a/r, ar, ar^3 & so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P, the product of two terms which are equidistant from the first and the last term, is constant and is equal to the product of first and last term. $\Rightarrow$ $T_k \cdot T_{n-k+1}$ = constant = a.l

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k} \cdot T_{r+k}$, $k < r, r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3.....a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2,.....\log a_n$ is an A.P. and vice-versa.

(i) If $a_1, a_2, a_3.....$ and $b_1, b_2, b_3.....$ are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3.....$ & $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}..........$ is also in G.P.

Ex. Find three numbers in G.P. having sum 19 and product 216.

Sol. Let the three numbers be $\dfrac{a}{r}$, a, ar

so $\quad a\left[\dfrac{1}{r}+1+r\right] = 19 \quad$ ....**(i)**

and $\quad a^3 = 216 \quad \Rightarrow \quad a = 6$

so from **(i)** $\quad 6r^2 - 13r + 6 = 0.$

$\Rightarrow \quad r = \dfrac{3}{2}, \dfrac{2}{3}$

Hence the three numbers are 4, 6, 9.

Ex. If a, b, c, d and p are distinct real numbers such that

$\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+cd\right)+\left(b^2+c^2+d^2\right)\le 0$ then which progression is suitable for a, b, c, d.

Sol. Here, the given condition $\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+ca\right)+b^2+c^2+d^2\le 0$

$\Rightarrow \left(ap-b\right)^2+\left(bp-c\right)^2+\left(cp-d\right)^2\le 0$

$\because$ a square can not be negative

$\therefore\ ap-b=0, bp-c=0, cp-d=0 \quad\Rightarrow\quad p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c} \quad\Rightarrow\quad$ a, b, c, d are in G.P.

Ex. Using G.P. express $0.\overline{3}$ and $1.2\overline{3}$ as $\frac{p}{q}$ form.

Sol. Let $x=0.\overline{3}=0.3333\ldots\ldots\ldots\ldots$

$=0.3+0.03+0.003+0.0003+\ldots\ldots\ldots\ldots$

$=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\ldots\ldots\ldots\ldots.$

$=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{9}=\frac{1}{3}.$

Let $y=1.2\overline{3}$

$=1.233333$

$=1.2+0.03+0.003+0.0003+\ldots\ldots\ldots\ldots$

$=1.2+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\ldots\ldots\ldots$

$=1.2+\frac{\frac{3}{10^2}}{1-\frac{1}{10}}=1.2+\frac{1}{30}=\frac{37}{30}.$

Harmonic Progression (H.P.)

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence $a_1, a_2, a_3, \ldots\ldots, a_n$ is an HP then $1/a_1, 1/a_2, \ldots\ldots\ldots, 1/a_n$ is an AP . Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots\ldots \frac{1}{a+(n-1)d}$

No term of any H.P. can be zero.

(a) Here we do not have the formula for the sum of the n terms of an H.P.. For H.P. whose first term is a and second term is b, the n^{th} term is $t_n=\frac{ab}{b+(n-1)(a-b)}$.

KEY POINTS

(i) If a, b, c are in H.P. $\Rightarrow\ b=\frac{2ac}{a+c}$ or $\frac{a}{c}=\frac{a-b}{b-c}$.

(ii) If a, b, c are in A.P. $\Rightarrow\ \frac{a-b}{b-c}=\frac{a}{a}$

(iii) If a, b, c are in G.P. $\Rightarrow\ \frac{a-b}{b-c}=\frac{a}{b}$

Ex. If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$, prove that a, b, c are in H.P, or b = a + c

Sol. We have $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$,

$\Rightarrow \quad \frac{a+c}{ac}+\frac{c-b+a-b}{(a-b)(c-b)} \qquad \Rightarrow \quad \frac{a+c}{ac}+\frac{(a+c)-2b}{ac-b(a+c)+b^2}=0$

Let $a+c=\lambda$

$\therefore \quad \frac{\lambda}{ac}+\frac{\lambda-2b}{ac-b\lambda+b^2}=0$

$\Rightarrow \quad \frac{ac\lambda-b\lambda^2+b^2\lambda+ac\lambda-2abc}{ac(ac-b\lambda+b^2)}=0$

$\Rightarrow \quad 2ac\lambda-b\lambda^2+b^2\lambda-2abc=0$

$\Rightarrow \quad 2ac(\lambda-b)-b\lambda(\lambda-b)=0 \qquad \Rightarrow \quad (2ac-b\lambda)(\lambda-b)=0$

$\Rightarrow \quad \lambda=b \text{ or } \lambda=\frac{2ac}{b}$

$\Rightarrow \quad a+c=b \text{ or } a+c=\frac{2ac}{b} \qquad (\because a+c=\lambda)$

$\Rightarrow \quad a+c=b \text{ or } b=\frac{2ac}{a+c}$

$\therefore$ a, b, c are in H.P. or a + c = b.

Ex. If m^{th} term of H.P. is n, while n^{th} term is m, find its $(m+n)^{th}$ term.

Sol. Given $T_m = n$ or $\frac{1}{a+(m-1)d}=n$; where a is the first term and d is the common difference of the corresponding A.P.

so $\quad a+(m-1)d=\frac{1}{n}$

and $\quad a+(n-1)d=\frac{1}{m} \qquad \Rightarrow \quad (m-n)d=\frac{m-n}{mn}$

or $\quad d=\frac{1}{mn} \qquad$ So $\quad a=\frac{1}{n}-\frac{(m-1)}{mn}=\frac{1}{mn}$

Hence $\quad T_{(m+n)}=\frac{1}{a+(m+n-1)d}=\frac{mn}{1+m+n-1}=\frac{mn}{m+n}$.

MEANS

(a) Arithmetic Mean (A.M.)

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c. So A.M. of a and c $=\frac{a+c}{2}=b$.

n-Arithmetic Means Between two Numbers

If a,b be any two given numbers & a, A_1, A_2, ..., A_n, b are in AP, then A_1, A_2,........A_n are the 'n' A.M's between a & b then. $A_1 = a + d$, $A_2 = a + 2d$,......., $A_n = a + nd$ or $b - d$, where $d=\frac{b-a}{n+1}$

$\Rightarrow \quad A_1=a+\frac{b-a}{n+1},\ A_2=a+\frac{2(b-a)}{n+1}$,.......

ETOOS KEY POINTS

Sum of n A.M's inserted between a & b is equal to n times the single A.M. between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Ex. Insert 20 A.M. between 2 and 86.

Sol. Here 2 is the first term and 86 is the 22nd term of A.P. so 86 = 2 + (21)d

$\Rightarrow$ d = 4

so the series is 2, 6, 10, 14,......., 82, 86

$\therefore$ required means are 6, 10, 14,...,82.

Ex. Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

Sol. Let a and b be two numbers and 2n A.M.s are inserted between a and b, then

$$\frac{2n}{2}(a+b) = 2n+1.$$

$$n\left(\frac{13}{6}\right) = 2n+1. \qquad \left[\text{given } a+b=\frac{13}{6}\right]$$

$\Rightarrow$ n = 6.

$\therefore$ Number of means = 12.

(b) Geometric Mean (G.M.)

If a, b, c are in G.P., then b is the G.M. between a & c, $b^2 = ac$. So G.M. of a and c = $\sqrt{ac} = b$

n-Geometric Means Between two Numbers

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in G.P. Then G_1, G_2, G_3 ,.......G_n are 'n' G.Ms between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ..., $G_n = a(b/a)^{n/n+1}$

= ar, = ar^2, ... = ar^n = b/r, where $r = (b/a)^{1/n+1}$

KEY POINTS

The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$, where G is the single G.M. between a & b.

Ex. Insert 4 G.M.s between 2 and 486.

Sol. Common ratio of the series is given by $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = (243)^{1/5} = 3$

Hence four G.M.s are 6, 18, 54, 162.

(c) Harmonic Mean (H.M.)

If a, b, c are in H.P., then b is H.M. between a & c. So H.M. of a and c = $\frac{2ac}{a+c}$ = b.

n-Harmonic Means Between two Numbers

a, H_1, H_2, H_3,........, H_n, b $\rightarrow$ H.P

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots\ldots, \frac{1}{H_n}, \frac{1}{b} \rightarrow A.P.$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \quad \Rightarrow \quad D = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$$

$$\frac{1}{H_n} = \frac{1}{a} + n\left(\frac{\frac{1}{b} - \frac{1}{a}}{n+1}\right)$$

Ex. Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Sol. Let 'd' be the common difference of corresponding A.P..

So $\quad d = \frac{\frac{13}{2} - \frac{3}{2}}{5} = 1.$

$\therefore \quad \frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2}$ or $H_1 = \frac{2}{5}$

$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2}$ or $H_2 = \frac{2}{7}$

$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2}$ or $H_3 = \frac{2}{9}$

$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2}$ or $H_4 = \frac{2}{11}$.

RELATION BETWEEN A.M. , G.M. , H.M.

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.) and $A \geq G \geq H$.

A.M. ≥ G.M.≥ H.M.

Let a_1, a_2, a_3, ...a_n be n positive real numbers, then we define their

A.M. = $\frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}$, their

G.M. = $(a_1 a_2 a_3 \ldots\ldots\ldots a_n)^{1/n}$ and their

H.M. = $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}}$.

It can be shown that

A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff

$a_1 = a_2 = a_3 = \ldots\ldots\ldots\ldots = a_n$

Ex. The A.M. of two numbers exceeds the G.M. by $\frac{3}{2}$ and the G.M. exceeds the H.M. by $\frac{6}{5}$; find the numbers.

Sol. Let the numbers be a and b, now using the relation

$$G^2 = AH = \left(G+\frac{3}{2}\right)\left(G-\frac{6}{5}\right) = G^2 + \frac{3}{10}G - \frac{9}{5}$$

$\Rightarrow$ $G = 6$

i.e. $ab = 36$

also $a + b = 15$

Hence the two numbers are 3 and 12.

Ex. If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Sol. Let a, b, g be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $a_1 = -a, a_2 = -b, a_3 = -g$ $\Rightarrow$ $a_1 + a_2 + a_3 = \frac{a}{2}$

$a_1a_2 + a_2a_3 + a_3a_1 = \frac{b}{2}$

$a_1a_2a_3 = 2$

Applying AM $\geq$ GM, we have

$$\frac{\alpha_1 + \alpha_2 + \alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \quad \Rightarrow \quad a \geq 6 \times 2^{1/3}$$

Also $$\frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \quad \Rightarrow \quad b\ ^3\ 6 \times 4^{1/3}$$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Ex. If a, b, c > 0, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

Sol. Using the relation A.M. $\geq$ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right)^{\frac{1}{3}} \quad \Rightarrow \quad \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

ARITHMETICO - GEOMETRIC SERIES

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series, e.g. $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots$

Here 1, 3, 5, ... are in A.P. & 1, x, x^2, x^3 .... are in G.P.

(a) Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots\ldots + [a + (n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]\ r^n}{1-r}$, $r \neq 1$

(b) Sum of n terms of an Arithmetico-Geometric Series when n $\to \infty$

If $0 < |r| < 1$ & $n \to \infty$, then $\lim_{n\to\infty} r^n = 0$, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Ex. Find the sum of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots$ to n terms.

Sol. Let $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots + \frac{3n-2}{5^{n-1}}$ .....(i)

$\left(\frac{1}{5}\right) S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots\ldots. + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$ .....(ii)

(i) – (ii) ⇒

$$\frac{4}{5} S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots\ldots. + \frac{3}{5^{n-1}} - \frac{3n-2}{5^n}.$$

$$\frac{4}{5} S = 1 + \frac{\frac{3}{5}\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{1-\frac{1}{5}} - \frac{3n-2}{5^n} = 1 + \frac{3}{4} - \frac{3}{4} \times \frac{1}{5^{n-1}} - \frac{3n-2}{5^n}$$

$$= \frac{7}{4} - \frac{12n+7}{4.5^n} \qquad \therefore \qquad S = \frac{35}{16} - \frac{(12n+7)}{16\,.\,5^{n-1}}$$

Ex. Find the sum of series $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$.

Sol. Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ldots\ldots\ldots. \infty$

On subtraction, we get

$$S(1+x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\ldots\ldots \infty$$

$$-S(1+x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\ldots\ldots \infty$$

On subtraction, we get

$$S(1+x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\ldots\ldots \infty$$

$$= 4 - x + 2x^2(1 - x + x^2 - \ldots\ldots\ldots\infty) = 4 - x + \frac{2x^2}{1+x} = \frac{4+3x+x^2}{1+x}$$

$$S = \frac{4+3x+x^2}{(1+x)^3}$$

Ex. Evaluate $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots$ upto infinity, where $|x| < 1$.

Sol. Let $S = 1 + 2x + 3x^2 + 4x^3 + \ldots..$ .....(i)

$xS = x + 2x^2 + 3x^3 + \ldots\ldots\ldots$ .....(ii)

(i) - (ii) ⇒ $(1-x)\,S = 1 + x + x^2 + x^3 + \ldots\ldots\ldots.$

or $S = \frac{1}{(1-x)^2}$

SIGMA NOTATIONS (Σ)

Properties

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(b) $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$; where k is a constant.

Some Results

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

(e) $\sum_{r=1}^{n} (2r-1) = n^2$ (sum of first n odd natural numbers)

(f) $\sum_{r=1}^{n} 2r = n(n+1)$ (sum of first n even natural numbers)

KEY POINTS

If n^{th} term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants, then sum of n terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$

Ex. Find the sum of the series to n terms whose general term is 2n + 1.

Sol. $\Sigma_n = \Sigma T_n = \Sigma(2n+1)$

$= 2\Sigma n + \Sigma 1$

$= \frac{2(n+1)\,n}{2} + n = n^2 + 2n$

Ex. Sum up to 16 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ is

Sol. $t_n = \frac{1^3+2^3+3^3+....+n^3}{1+3+5+....(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4} = \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4}$

$\therefore \quad S_n = \Sigma t_n = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1 \quad = \frac{1}{4}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} + \frac{1}{4}.n$

$\therefore \quad S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$

Ex. Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Sol. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j$

$= \sum_{i=1}^{n} \frac{i\,(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$

$= \frac{1}{2}\left[\frac{n\,(n+1)\,(2n+1)}{6} + \frac{n\,(n+1)}{2}\right]$

$= \frac{n\,(n+1)}{12}\,[2n+1+3] = \frac{n\,(n+1)\,(n+2)}{6}$.

METHOD OF DIFFERENCE

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier.

So we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.
If $T_1, T_2, T_3, ..., T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, ...$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Method of Difference for Finding n^{th} Term

Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1, u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ....**(i)**

$S = \quad u_1 + u_2 + ... + u_{n-1} + u_n$ ....**(ii)**

(i) – (ii) $\Rightarrow u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n.

So sum of series $S = \sum_{r=1}^{n} u_r$

Case - I

(a) If difference series are in A.P., then

Let $T_n = an^2 + bn + c$, where a, b, c are constant

(b) If difference of difference series are in A.P.

Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant

Case - II

(a) If difference are in G.P., then

Let $T_n = ar^n + b$, where r is common ratio & a, b are constant

(b) If difference of difference are in G.P., then

Let $T_n = ar^n + bn + c$, where r is common ratio & a, b, c are constant

Determine constant by putting n = 1, 2, 3 .... n and putting the value of T_1, T_2, T_3 ...... and sum of series $(S_n) = \sum T_n$

Ex. Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Sol. Let $S = 3 + 7 + 13 + 21 + ... + T_n$ .....**(i)**

$S = \quad 3 + 7 + 13 + ... + T_{n-1} + T_n$ .....**(ii)**

(i) – (ii)

$\Rightarrow T_n = 3 + 4 + 6 + 8 + ... + (T_n - T_{n-1})$

$= 3 + \frac{n-1}{2}[8 + (n-2)2]$

$= 3 + (n-1)(n+2)$

$= n^2 + n + 1$

Hence $S = \Sigma(n^2 + n + 1)$

$= \sum n^2 + \sum n + \sum 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n \quad = \frac{n}{3}(n^2 + 3n + 5)$

Method of Difference for Finding s_n

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$$t_1 = f(1) - f(0),$$
$$t_2 = f(2) - f(1),$$
$$\vdots \quad \vdots \quad \vdots$$
$$t_n = f(n) - f(n-1)$$
$$\Rightarrow \quad S_n = f(n) - f(0)$$

Ex. Find the sum of n-terms of the series 1.2 + 2.3 + 3.4 + ...

Sol. Let T_r be the general term of the series

So $T_r = r(r+1)$.

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r+2)-(r-1)]$

So $T_r = \frac{r}{3}(r+1)[(r+2)-(r-1)]$

$$= \frac{1}{3}[r(r+1)(r+2) - (r-1)r(r+1)].$$

Let $f(r) = \frac{1}{3} r(r+1)(r+2)$

So $T_r = [f(r) - f(r-1)]$.

Now $S = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$$T_1 = \frac{1}{3}[1.2.3 - 0]$$
$$T_2 = \frac{1}{3}[2.3.4 - 1.2.3]$$
$$T_3 = \frac{1}{3}[3.4.5 - 2.3.4]$$
$$\vdots$$
$$T_n = \frac{1}{3}[n(n+1)(n+2) - (n-1)n(n+1)]$$

$$\therefore \quad S = \frac{1}{3} n(n+1)(n+2)$$

Hence sum of series is f(n) – f(0).

Ex. Find the nth term and the sum of n term of the series $2 + 12 + 36 + 80 + 150 + 252 + ...$

Sol. Let $S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_n$ .....(i)

$S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_{n-1} + T_n$ .....(ii)

(i) – (ii)

$\Rightarrow$ $T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ .....(iii)

$T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ .....(iv)

(iii) – (iv)

$\Rightarrow$ $T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ...$

$= \frac{n}{2}[4 + (n-1)6] = n[3n - 1] \Rightarrow T_n - T_{n-1} = 3n^2 - n$

$\therefore$ general term of given series is $\Sigma(T_n - T_{n-1}) = \Sigma(3n^2 - n) = n^3 + n^2$.

Hence sum of this series is

$$S = \Sigma n^3 + \Sigma n^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{12}(3n^2 + 7n + 2)$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+1)$$

Ex. If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Sol. $T_n = S_n - S_{n-1}$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8} = \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \quad .....(i)$$

Let $V_n = \frac{1}{n(n+1)}$

$\therefore$ $\frac{1}{T_n} = V_n - V_{n+1}$

Putting $n = 1, 2, 3, ... n$

$$\Rightarrow \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + ... + \frac{1}{T_n} = (V_1 - V_{n+1})$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{T_r} = \frac{n^2 + 3n}{2(n+1)(n+2)}$$

TIPS AND TRICKS

1. Arithmetic Progression (AP)

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the **Common Difference**. If 'a' is the first term & 'd' is the common difference, then AP can be written as

$a, a + d, a + 2d, \ldots\ldots\ldots\ldots a + (n - 1)d, \ldots\ldots\ldots\ldots$

(a) n^{th} term of this AP $T_n = a + (n - 1)d$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $Sn = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]$ where ℓ is the last term.

(c) Also nth term $T_n = S_n - S_{n-1}$

Note

(i) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference twice the coefficient of n^2 i.e. 2A

(ii) n^{th} term of an A.P. is of the form $A_n + B$ i.e. a linear expression in n, in such case the coefficient of n is the common difference of A.P. i.e. A

(iii) Three numbers is AP can be taken as $a - d, a, a + d$; four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$ five numbers in AP are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in AP are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) If for A.P. pth term is q, q^{th} term is p, then r^{th} term is $= p + q - r$ & $(P + q)^{th}$ term is 0.

2. Geometric Progression (GP)

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by the immediately previous term.

Therefore $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots$ is a GP with 'a' as the first term & 'r' as common ratio.

(a) nth term $T_n = a\,r^{n-1}$

(b) Sum of the first n terms $S_n = \frac{a(r^n - 1)}{r - 1}$, if $r \neq 1$

(c) Sum of infinite GP when $|r| < 1$ & $n \to \infty$, $r^n \to 0$ $S_\infty = \frac{a}{1-r}; |r| < 1$

(d) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r3, a/r, ar, ar3 & so on.

(e) If a, b, c are in GP $\Rightarrow b^2 = ac \Rightarrow$ loga, logb, logc, are in A.P.

3. Harmonic Progression (HP)

A sequence is said to HP if the reciprocals of its terms are in AP. If the sequence $a_1, a_2, a_3 \ldots\ldots, a_n$ is an HP then $1/a_1, 1/a_2, \ldots\ldots\ldots 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots \frac{1}{a+(n-1)d}$

Note : No term of any H.P can be zero. If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

4. Means

(a) Arithmetic Mean (AM)

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

n-Arithmetic Means Between two Numbers

If a, b are any two given numbers & a, $A_1, A_2, \ldots\ldots A_n$, b are in AP then $A_1, A_2, \ldots. A_n$ are the n AM`s between a & b, then

$A_1 = a + d, A_2 = a + 2d, \ldots\ldots, A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note Sum of n AM`s inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

(b) Geometric Mean (GM)

If a, b, c are in GP, b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$

n-Geometric Means Between two Numbers

If a, b are two given positive numbers & a, $G_1, G_2, \ldots. G_n$, b are in GP then $G_1, G_2, G_3, \ldots. G_n$ are n GMs between a & b. $G_1 = ar, G_2 = ar_2, \ldots\ldots G_n = ar_n$, where $r = (b/a)1/^{n+1}$

Note The product of n GMs between a & b i.e. $\prod_{r=1}^{n} G_r = \mathbf{(G)^n}$ where G is the single GM between a & b

(c) Harmonic Mean (HM)

If a, b, c are in HP, then b is HM between a & c, then $b = \frac{2ac}{a+c}$.

Important Note

(i) If A, G, H are respectively AM, GM, HM between two positive number a & b then

(a) $G^2 = AH$ (A, G, H constitute a GP)

(b) $A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let $a_1, a_2, \ldots\ldots a_n$ be n positive real numbers, then we define their arithmetic mean **(A)**, geometric mean **(G)** and harmonic mean **(H)** as $A = \frac{a_1 + a_2 + \ldots. + a_n}{n}$, $G = (a_1, a_2 \ldots\ldots a_n)^{1/n}$ and $H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots. + \frac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

5. Arithmetico – Geometric Series

Sum of First n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a+d)r + (a+2d)r^2 + ...+ [a+(n-1)d]r^{n-1}$ then $S_n = \dfrac{a}{1-r} + \dfrac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \dfrac{\left[a+(n-1)d\right]r^n}{1-r}, r \neq 1$

Sum to Infinity

If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

6. Sigma Notations

Theorems

(a) $\sum\limits_{r=1}^{n}(a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r$ (b) $\sum\limits_{r=1}^{n} ka_r = k\sum\limits_{r=1}^{n} a_r$ (c) $\sum\limits_{r=1}^{n} k = nk$; where k is a constant.

7. Results

(a) $\sum\limits_{r=1}^{n} r = \dfrac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum\limits_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum\limits_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4} = \left[\sum\limits_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum\limits_{r=1}^{n} r^4 = \dfrac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

8. Method of Difference

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier. So we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.

If $T_1, T_2, T_3, ..., T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, ...$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.